The Gamma Function

Gary Schurman, MBE, CFA

April, 2016

The Gamma Distribution is a distribution of waiting times between Poisson distributed events. For example we may want to know the probability that a certain event will occur within the next 48 hours. The probability distributions related to the Gamma Distribution are the Beta, Exponential, Chi-Squared and Erlang distributions.

All of the probability distributions noted above use the gamma function. The gamma function is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. The motivation behind the gamma function was to find a smooth curve that connects the points (x, y) given by $y = (x-1)!$ at the positive integer values for x.

The Gamma Function Equation

If the variable α is a positive integer greater than zero then the gamma function of α is...

$$\Gamma(\alpha) = (\alpha - 1)! \text{ ...where... } \{\alpha \,|\, \alpha \in \mathbb{Z}, \alpha > 0\} \tag{1}$$

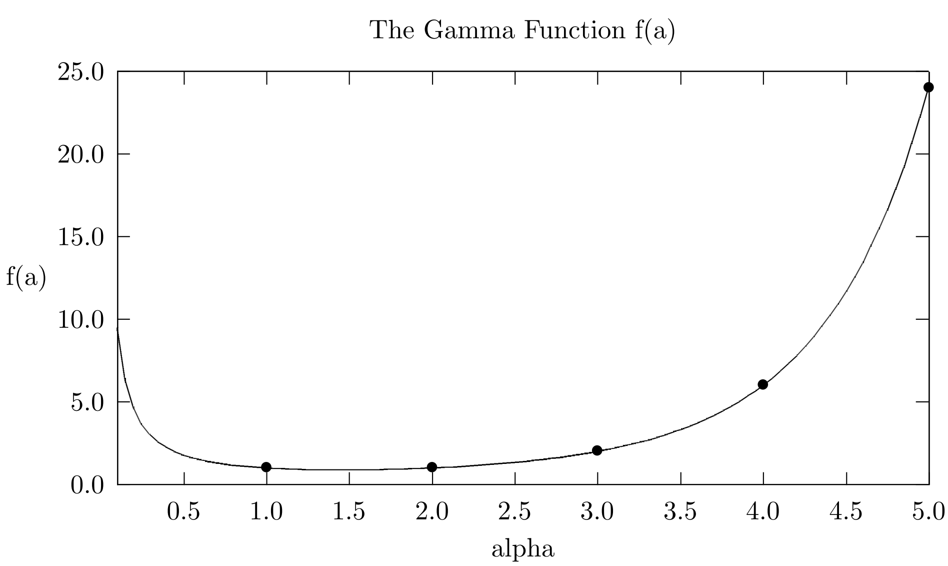
For the more general case where the variable α is a positive real number greater than zero then the gamma function of α is...

$$\Gamma(\alpha) = \int\limits_0^\infty u^{\alpha-1} \,\mathrm{Exp}\left\{-u\right\} \delta u \text{ ...where... } \{\alpha \,|\, \alpha \in \mathbb{R}, \alpha > 0\} \tag{2}$$

In Excel the gamma function of α is represented by the following Excel function...

$$\Gamma(\alpha) = \mathrm{EXP}(\mathrm{GAMMALN}(\alpha)) \text{ ...where... } \alpha > 0 \tag{3}$$

The following graph charts the gamma function $f(\alpha)$ over the real number interval $[\alpha > 0, \alpha \leq 5]$...



Note the following...

1 The line goes through the point 1, (1-0)!
2 The line goes through the point 2, (2-0)!
3 The line goes through the point 3, (3-0)!
4 The line goes through the point 4, (4-0)!
5 The line goes through the point 5, (5-0)!
6 As n goes to zero from the right gamma(n) goes to infinity
7 As n goes to infinity from the left gamma(n) goes to infinity

The Solution To The Gamma Function Integral

We want to derive the numerical solution to the gamma function integral. To that end we will make the following definition...

$$\mathrm{Exp}\left\{-u\right\} = \lim_{n\to\infty}\left(1-\frac{u}{n}\right)^n \tag{4}$$

Using Equations (2) and (4) above we will define the function $\Gamma(\alpha, n)$ as follows...

$$\Gamma(\alpha,n) = \int\limits_{u=0}^{u=n} u^{\alpha-1}\left(1-\frac{u}{n}\right)^n \delta u \text{ ...such that... } \Gamma(\alpha) = \lim_{n\to\infty}\Gamma(\alpha,n) \tag{5}$$

We will make the following definition...

$$\text{if... } u = \lambda\, n \text{ ...then... } \lambda = \frac{1}{n}\,u \text{ ...and... } \frac{\delta\lambda}{\delta u} = \frac{1}{n} \text{ ...and... } \delta u = n\,\delta\lambda \tag{6}$$

Using the definitions in Equation (6) above we can rewrite Equation (5) above as follows...

$$\Gamma(\alpha,n) = \int\limits_{u=0}^{u=n} (\lambda\, n)^{\alpha-1}\left(1-\frac{\lambda\, n}{n}\right)^n n\,\delta\lambda = \int\limits_{u=0}^{u=n} \lambda^{\alpha-1}\, n^{\alpha-1}\Big(1-\lambda\Big)^n n\,\delta\lambda = \int\limits_{u=0}^{u=n} n^{\alpha}\lambda^{\alpha-1}\Big(1-\lambda\Big)^n \delta\lambda \tag{7}$$

Using the definitions in Equation (6) above and changing the integrand in Equation (7) above from u to λ we have the following revised bounds of integration...

$$\text{Upper bound} = \frac{1}{n}\times n = 1 \text{ ...and... Lower bound} = \frac{1}{n}\times 0 = 0 \text{ ...given that... } \lambda = \frac{1}{n}\,u \tag{8}$$

Using the revised bounds of integration per Equation (8) above we can rewrite Equation (7) above as...

$$\Gamma(\alpha,n) = n^{\alpha}\int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha-1}\Big(1-\lambda\Big)^n \delta\lambda \tag{9}$$

Using Appendix Equations (18), (19) and (20) below the solutions to Equation (9) above when $n=1$, $n=2$ and $n=3$, respectively, are...

$$\begin{aligned}
\Gamma(\alpha,1) &= \frac{1}{\alpha\,(\alpha+1)} \text{ ...when... } n=1\\
\Gamma(\alpha,2) &= 2^{\alpha}\left(\frac{2}{\alpha\,(\alpha+1)\,(\alpha+2)}\right) \text{ ...when... } n=2\\
\Gamma(\alpha,3) &= 3^{\alpha}\left(\frac{6}{\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)}\right) \text{ ...when... } n=3
\end{aligned} \tag{10}$$

Noting the pattern in Equation (10) above we can make the following statement...

$$\Gamma(\alpha) = \lim_{n\to\infty}\Gamma(\alpha,n) = \lim_{n\to\infty} n^{\alpha}\, n!\Big/\Pi_{i=0}^{n}(\alpha+i) \tag{11}$$

Since calculating the factorial of a large number is problematic (number becomes so large that the computer and/or calculator cannot handle it) we may opt to calculate the natural log of the gamma function instead. Using Equation (11) above the log of the gamma function is...

$$\ln\Bigg(\Gamma(\alpha)\Bigg)=\lim_{n\to\infty}\alpha\,\ln(n)+\sum_{i=1}^{n}\ln(i)-\sum_{i=0}^{n}\ln(\alpha+i)\;\text{...such that...}\;\Gamma(\alpha)=\text{Exp}\Bigg\{\ln\Bigg(\Gamma(\alpha)\Bigg)\Bigg\} \tag{12}$$

Integration By Parts

Integration by parts gives us the following equation...

$$\int g(u)\,\delta f(u)=f(u)\,g(u)-\int f(u)\,\delta g(u) \tag{13}$$

We will define the function $f(u)$ as follows...

$$\text{if...}\;f(u)=\frac{1}{\alpha}\,u^{\alpha}\;\text{...then...}\;\frac{\delta f(u)}{\delta u}=u^{\alpha-1}\;\text{...such that...}\;\delta f(u)=u^{\alpha-1}\,\delta u \tag{14}$$

We will define the function $f(u)$ as follows...

$$\text{if...}\;g(u)=\text{Exp}\Bigg\{-u\Bigg\}\;\text{...then...}\;\frac{\delta g(u)}{\delta u}=-\text{Exp}\Bigg\{-u\Bigg\}\;\text{...such that...}\;\delta g(u)=-\text{Exp}\Bigg\{-u\Bigg\}\,\delta u \tag{15}$$

Using the definitions in Equations (14) and (15) above we can rewrite Equation (2) above as...

$$\Gamma(\alpha)=\int g(u)\,\delta f(u)=f(u)\,g(u)-\int f(u)\,\delta g(u)=\frac{1}{\alpha}\,u^{\alpha}\,\text{Exp}\Bigg\{-u\Bigg\}+\frac{1}{\alpha}\int u^{\alpha}\,\text{Exp}\Bigg\{-u\Bigg\}\,\delta u \tag{16}$$

Using Appendix Equations (21) and (23) below the solution to Equation (16) above is...

$$\Gamma(\alpha)=0+\frac{1}{\alpha}\,\Gamma(\alpha+1)\;\text{...such that...}\;\alpha\,\Gamma(\alpha)=\Gamma(\alpha+1) \tag{17}$$

Appendix

A. The solution to Equation (9) above when $n=1$ is...

$$\begin{aligned}\Gamma(\alpha,1)&=[1]^{\alpha}\int\limits_{\lambda=0}^{\lambda=1}\lambda^{\alpha-1}\Bigg(1-\lambda\Bigg)^{[1]}\delta\lambda\\&=\int\limits_{\lambda=0}^{\lambda=1}\lambda^{\alpha-1}\,\delta\lambda-\int\limits_{\lambda=0}^{\lambda=1}\lambda^{\alpha}\,\delta\lambda\\&=\frac{1}{\alpha}\,\lambda^{\alpha}\Bigg[_{\lambda=0}^{\lambda=1}-\frac{1}{\alpha+1}\,\lambda^{\alpha+1}\Bigg[_{\lambda=0}^{\lambda=1}\\&=\frac{1}{\alpha}-\frac{1}{\alpha+1}\\&=\frac{1}{\alpha\,(\alpha+1)}\end{aligned} \tag{18}$$

B. The solution to Equation (9) above when $n = 2$ is...

$$\begin{aligned}\Gamma(\alpha, 2) &= [2]^\alpha \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha-1} \bigg(1-\lambda\bigg)^{[2]} \delta\lambda \\ &= 2^\alpha \left(\int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha-1}\,\delta\lambda - 2 \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha}\,\delta\lambda + \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha+1}\,\delta\lambda \right) \\ &= 2^\alpha \left(\frac{1}{\alpha}\,\lambda^\alpha \Bigg[_{\lambda=0}^{\lambda=1} - \frac{2}{\alpha+1}\,\lambda^{\alpha+1} \Bigg[_{\lambda=0}^{\lambda=1} + \frac{1}{\alpha+2}\,\lambda^{\alpha+2} \Bigg[_{\lambda=0}^{\lambda=1} \right) \\ &= 2^\alpha \left(\frac{1}{\alpha} - \frac{2}{\alpha+1} + \frac{1}{\alpha+2} \right) \\ &= 2^\alpha \left(\frac{2}{\alpha\,(\alpha+1)\,(\alpha+2)} \right) \end{aligned} \tag{19}$$

C. The solution to Equation (9) above when $n = 3$ is...

$$\begin{aligned}\Gamma(\alpha, 3) &= [3]^\alpha \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha-1} \bigg(1-\lambda\bigg)^{[3]} \delta\lambda \\ &= 3^\alpha \left(\int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha-1}\,\delta\lambda - 3 \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha}\,\delta\lambda + 3 \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha+1}\,\delta\lambda - \int\limits_{\lambda=0}^{\lambda=1} \lambda^{\alpha+2}\,\delta\lambda \right) \\ &= 3^\alpha \left(\frac{1}{\alpha}\,\lambda^\alpha \Bigg[_{\lambda=0}^{\lambda=1} - \frac{3}{\alpha+1}\,\lambda^{\alpha+1} \Bigg[_{\lambda=0}^{\lambda=1} + \frac{3}{\alpha+2}\,\lambda^{\alpha+2} \Bigg[_{\lambda=0}^{\lambda=1} - \frac{1}{\alpha+3}\,\lambda^{\alpha+3} \Bigg[_{\lambda=0}^{\lambda=1} \right) \\ &= 3^\alpha \left(\frac{1}{\alpha} - \frac{3}{\alpha+1} + \frac{3}{\alpha+2} - \frac{1}{\alpha+3} \right) \\ &= 3^\alpha \left(\frac{6}{\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)} \right) \end{aligned} \tag{20}$$

D. The solution to the following equation is...

$$\frac{1}{\alpha}\,u^\alpha\,\mathrm{Exp}\bigg\{-u\bigg\} \Bigg[_0^\infty = 0 - 0 = 0 \tag{21}$$

Because...

$$\lim_{u\to\infty} \frac{1}{\alpha}\,u^\alpha\,\mathrm{Exp}\bigg\{-u\bigg\} = 0 \ \text{ ...and... } \frac{1}{\alpha}\,(0)^\alpha\,\mathrm{Exp}\bigg\{-(0)\bigg\} = 0 \tag{22}$$

E. Using Equation (2) above the solution to the following integral is...

$$\int\limits_{u=0}^{u=\infty} u^\alpha\,\mathrm{Exp}\bigg\{-u\bigg\}\,\delta u = \Gamma(\alpha+1) \tag{23}$$